MATH 31B, LECTURE 1
MIDTERM 2
MAY 18, 2012

Name: Solutions

UID:________________________

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Discussion meets: (circle one) Tuesday Thursday

Instructions: The exam is closed-book, closed-notes. Calculators are not permitted. Answer each question in the space provided. If the question is in several parts, carefully label the answer to each part. Do all of your work on the examination paper; scratch paper is not permitted. If you continue a problem on the back of the page, please write "continued on back".

Each problem is worth 20 points.

Problem	Score
1	
2	
3	
4	
5	
Total	

Problem 1:

(a) Evaluate the indefinite integral: $\int \tan^2 x \sec^6 x \, dx$

(b) Evaluate the indefinite integral: $\int \frac{\arctan x}{(1+x^2)^{3/2}} \, dx.$

Solution:

(a) Use the identity $\sec^2 x = (1+\tan^2 x)$ and then substitute $u = \tan x$:

$$\int \tan^2 x \sec^6 x \, dx = \int \tan^2 x (1+\tan^2 x)^2 \sec^2 x \, dx = \int u^2(1+u^2)^2 \, du$$
$$= \frac{u^7}{7} + \frac{2u^5}{5} + \frac{u^3}{3} + C = \frac{\tan^7 x}{7} + \frac{2\tan^5 x}{5} + \frac{\tan^3 x}{3} + C.$$

(b) Make the subsitute $x = \tan\theta$, $dx = \sec^2\theta \, d\theta$:

$$\int \frac{\arctan x}{(1+x^2)^{3/2}} \, dx = \int \frac{\theta}{(1+\tan^2\theta)^{3/2}} \sec^2\theta \, d\theta = \int \theta \cdot \cos\theta \, d\theta$$

Now use integration by parts with $u = \theta$, $dv = \cos\theta \, d\theta$:

$$\int \theta \cdot \cos\theta \, d\theta = \theta \cdot \sin\theta - \int \sin\theta \, d\theta = \theta \cdot \sin\theta + \cos\theta + C.$$

Now solve $\sin\theta = \frac{x}{\sqrt{x^2+1}}$ and $\cos\theta = \frac{1}{\sqrt{x^2+1}}$ so that

$$\int \frac{\arctan x}{(1+x^2)^{3/2}} \, dx = \frac{x \cdot \arctan x + 1}{\sqrt{x^2+1}} + C.$$

Problem 2:

(a) Evaluate the improper integral: $\int_1^\infty \frac{x+1}{x^4+x^2} \, dx.$

(b) Use the comparison test to determine if the improper integral converges: $\int_0^\infty \frac{2x\cos^2 x}{7x+5x^3} \, dx$

Solution:

(a) Use partial fractions:

$$\frac{x+1}{x^4+x^2} = \frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{x^2+1} = \frac{A(x^2+1) + B(x^3+x) + Cx^3 + Dx^2}{x^4+x^2}.$$

This gives the system of equations:

$$A + D = 0$$
$$B + C = 0$$
$$B = 1$$
$$A = 1$$

Solving we find $A = 1, B = 1, C = -1, D = -1$. So

$$\int_1^\infty \frac{x+1}{x^4+x^2}\,dx = \int_1^\infty \left(\frac{1}{x^2} + \frac{1}{x} - \frac{x+1}{x^2+1}\right)dx = \lim_{t\to\infty}\left(\frac{-1}{x} + \ln x - \frac{\ln(x^2+1)}{2} - \arctan x\right)\Bigg|_{x=1}^{t}$$

$$= \lim_{t\to\infty}\left(\left(\ln t - \frac{\ln(t^2+1)}{2}\right) - \frac{\pi}{2}\right) - \left(-1 - \frac{\ln 2}{2} - \frac{\pi}{4}\right)$$

Now determine $\lim_{t\to\infty}(\ln t - \frac{\ln(t^2+1)}{2})$:

$$\lim_{t\to\infty}\left(\ln t - \frac{\ln(t^2+1)}{2}\right) = \lim_{t\to\infty}\ln\left(\frac{t}{\sqrt{t^2+1}}\right) = \ln 1 = 0.$$

So combining the formulas above we get

$$\int_1^\infty \frac{x+1}{x^4+x^2}\,dx = 1 + \frac{\ln 2}{2} - \frac{\pi}{4}$$

(b) We have $0 \leq \cos^2 x \leq 1$ for all x, which implies

$$\frac{2x\cos^2 x}{7x+5x^3} \leq \frac{2x}{7x+5x^3} = \frac{2}{7+5x^2} \leq \frac{1}{x^2}$$

for $x \geq 1$. Since

$$\int_1^\infty \frac{1}{x^2}\,dx$$

is convergent by the p-test, $\displaystyle\int_1^\infty \frac{2x\cos^2 x}{7x+5x^3}\,dx$ converges by the comparison test. Since $\frac{2x\cos^2 x}{7x+5x^3} = \frac{2\cos^2 x}{7+5x^2}$ is continuous on $[0,1]$, $\displaystyle\int_0^\infty \frac{2x\cos^2 x}{7x+5x^3}\,dx$ is convergent as well.

Problem 3:

(a) Find M_4 and S_4 for $\displaystyle\int_2^4 e^x\,dx$ (you do not need to simplify your expressions).

(b) Does M_4 give an overestimate or underestimate of the integral?

(c) Compute the error bound for S_4.

Solution:

(a) We have $\Delta x = \frac{4-2}{4} = \frac{1}{2}$.

$$M_4 = \frac{1}{2}\left(e^{2.25} + e^{2.75} + e^{3.25} + e^{3.75}\right)$$

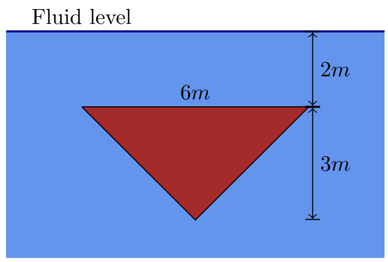
$$S_4 = \frac{1}{3\cdot 2}\left(e^2 + 4e^{2.5} + 2e^3 + 4e^{3.5} + e^4\right)$$

(b) Since e^x is convex up on this interval, M_4 gives an underestimate.

(c) For $f(x) = e^x$ we have $f^{(4)}(x) = e^x$ to the maximum value K_4 of $f^{(4)}(x)$ on $[2,4]$ is e^4. We then have

$$|Error(S_4)| \leq \frac{K_4(4-2)^5}{180\cdot 4^4} = \frac{e^4\cdot 32}{180\cdot 256}.$$

Problem 4: Find the pressure on the triangular plate in the figure below, submerged in a fluid of density $\rho = 200\ kg/m^3$. The top of the plate is parallel with the surface of the fluid, and is at a depth of 2 meters below the surface. Assume $g = 9.8\ m/s^2$.



Solution: Let $f(y)$ be the width of the triangle at depth y, $2 \le y \le 5$. It is clear that $f(y)$ is a linear function, so since $f(2) = 6$ and $f(5) = 0$ we conclude the slope is $\frac{0-6}{5-2} = -2$ and hence

$$f(y) = -2y + 10.$$

So the force is

$$(\rho g) \int_2^5 y f(y)\, dy = (200 \cdot 9.8) \int_2^5 y \cdot (-2y + 10)\, dy = (200 \cdot 9.8) \cdot \left(\frac{-2y^3}{3} + 5y^2 \right) \Big|_{y=2}^{5}$$

$$= 1960 \cdot \left(\left(\frac{-2 \cdot 125}{3} + 5 \cdot 25\right) - \left(\frac{-2 \cdot 8}{3} + 5 \cdot 4\right) \right) = 1960 \cdot 27 = 52920$$

Problem 5:

(a) Find the Taylor polynomials $T_n(x)$ for $f(x) = \frac{1}{x}$, centered at $a = 1$.

(b) Compute the error bound for $|f(2) - T_n(2)|$.

(c) Find $|f(2) - T_{1000000}(2)|$.

Solution:

(a) Computing derivates you find $f^{(n)}(x) = (-1)^n n! \cdot x^{-(n+1)}$ so $f^{(n)}(1) = (-1)^n n!$. Plugging into the formula for the Taylor polynomials we find

$$T_n(x) = \sum_{j=0}^{n} (-1)^j (x-1)^j = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n (x-1)^n.$$

(b) The maximum value of $|f^{(n+1)}(u)| = |(n+1)! \cdot u^{-(n+1)}|$ on the interval $[1, 2]$ occurs at $u = 1$ where the value is $K = (n+1)!$. We then have

$$|f(2) - T_n(2)| \le \frac{K \cdot (2-1)^{(n+1)}}{(n+1)!} = \frac{(n+1)! \cdot 1}{(n+1)!} = 1.$$

(c) Compute $T_0(2) = 1$, $T_1(2) = 0$, $T_2(2) = 1$, $T_3(2) = 0$, etc. In general we have $T_n(2) = 1 - 1 + 1 - 1 + \cdots + (-1)^n$ is 1 or 0 depending on whether n is odd or even. So $T_{1000000}(2) = 1$ and hence $|f(2) - T_{1000000}(2)| = |1/2 - 1| = 1/2$. (Note that actually $|f(2) - T_n(2)| = 1/2$ for all values of n, since for the odd terms we have $|1/2 - 0| = 1/2$ as well).